M 340 Assignment #4 Solutions

In each of the following problems write out the governing differential equation, find the solution to the equation and make the required calculation.

1. If you invest $1000 at an interest rate of 5% per year and make regular monthly deposits of $150. What will be the value of your account after 30 years?

$$A'(t) = .05A(t) + D \quad A(0) = 1000 \quad D = 150 \times 12 = 1800$$
$$A(t) = Ce^{.05t} - \frac{1800}{.05} \quad C = 1000 + \frac{1800}{.05} = 37000$$
$$A(t) = 37000e^{.05t} - 36000$$
$$A(30) = 37000e^{1.5} - 36000 = \$129,820$$

2. If you borrow $25000 at an interest rate of 6% per year and make regular payments of $200 each month, how long will it take to pay off the loan? If you double the monthly payments, does that cut the length in half?

$$A'(t) = .06A(t) - P \quad A(0) = 25000 \quad P = 200 \times 12 = 2400$$
$$A(t) = Ce^{.06t} + \frac{2400}{.06}$$
$$C = 25000 - \frac{2400}{.06} = 25000 - 40000 = -15000$$
$$A(t) = -15000e^{.06t} + 40000$$
$$A(T) = 40000 - 15000e^{.06T} = 0$$
$$e^{.06T} = \frac{40}{15} = 2.67 \qquad .06T = \ln(2.67) = 0.982$$
$$T = .982/.06 = 16.36 yrs$$

If $P = 400 \times 12 = 4800$ then $C = 25000 - 80000 = -55000$

so $A(t) = 80000 - 55000e^{.06t}$

$$e^{.06T} = \frac{80}{55} = 1.45 \qquad .06T = \ln(1.45) = 0.372$$
$$T = .372/.06 = 6.2\ yrs$$

3. Write out the differential equation that models the chemical reaction $A + B \leftrightarrows C$, letting a and b denote the initial ammounts of chemicals A and B, respectively. If $x(t)$ denotes the ammount of chemical C at time t, sketch the four solution curves that correspond to initial conditions: $x(0) = 0,\ 0 < x(0) < a,\ a < x(0) < b,\ b < x(0)$.

$$x'(t) = \alpha(a - x)(b - x)$$

4. An electric circuit contains a resistance, $R,$ and a capacitance, $C,$ and is driven by an alternating current $E_0 \cos \Omega t.$ What is the amplitude of the response, $i(t),$ expressed in terms of R, C, Ω and E_0?

$$
\begin{aligned}
Ri(t) + \frac{1}{C}Q(t) &= E_0 \cos \Omega t, \\
R\, i'(t) + \frac{1}{C} i(t) &= -\Omega E_0 \sin \Omega t, \quad i(0) = 0 \\
i(t) &= a\, e^{-\frac{1}{CR}t} - \frac{C\Omega E_0 \sin t\Omega - C^2 R \Omega^2 E_0 \cos t\Omega}{C^2R^2\Omega^2 + 1} \\
i(0) &= a + \frac{E_0 C^2 R \Omega^2}{C^2R^2\Omega^2 + 1} = 0 \\
i(t) &= -\frac{E_0 C^2 R \Omega^2}{C^2R^2\Omega^2 + 1} e^{-t/RC} - E_0 C\Omega \frac{\sin t\Omega - RC\Omega \cos t\Omega}{C^2R^2\Omega^2 + 1} \\
i(t) &= \frac{E_0 C\Omega}{\sqrt{C^2R^2\Omega^2 + 1}} \frac{R\Omega C\ \cos t\Omega - \sin t\Omega}{\sqrt{C^2R^2\Omega^2 + 1}} - \frac{E_0 C^2 R \Omega^2}{C^2R^2\Omega^2 + 1} e^{-t/RC} \\
i(t) &= \frac{E_0 C\Omega}{\sqrt{C^2R^2\Omega^2 + 1}} [\cos\theta \cos \Omega t - \sin\theta \sin t\Omega] - \frac{E_0 C}{C^2R^2\Omega^2 + 1} e^{-t/RC} \\
\cos\theta &= \frac{R\Omega C}{\sqrt{C^2R^2\Omega^2 + 1}} \qquad \sin\theta = \frac{1}{\sqrt{C^2R^2\Omega^2 + 1}} \\
Amplitude &= \frac{E_0 C\Omega}{\sqrt{C^2R^2\Omega^2 + 1}}
\end{aligned}
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5.An electric circuit contains a resistance, $R,$ and an inductance, $L,$ and is driven by an alternating current $E_0 \cos \Omega t.$ What is the amplitude of the response, $i(t),$ expressed in terms of R, L, Ω and E_0?

$$
\begin{aligned}
L \frac{di}{dt} + Ri &= E_0 \cos \Omega t \qquad i(0) = 0 \\
i(t) &= \frac{RE_0 \cos t\Omega + L\Omega E_0 \sin t\Omega}{L^2\Omega^2 + R^2} + Ce^{-\frac{1}{L}Rt} \\
i(0) &= \frac{RE_0}{L^2\Omega^2 + R^2} + C = 0 \\
i(t) &= E_0 \frac{R \cos t\Omega + L\Omega \sin t\Omega}{\sqrt{L^2\Omega^2 + R^2}} - \frac{RE_0}{L^2\Omega^2 + R^2} e^{-\frac{1}{L}Rt} \\
i(t) &= \frac{E_0}{\sqrt{L^2\Omega^2 + R^2}} \left[\cos\theta\ \cos \Omega t - \sin\theta\ \sin \Omega t\right] - \frac{RE_0}{L^2\Omega^2 + R^2} e^{-\frac{1}{L}Rt} \\
\cos\theta &= \frac{R}{\sqrt{L^2\Omega^2 + R^2}} \qquad \sin\theta = \frac{-\Omega L}{\sqrt{L^2\Omega^2 + R^2}} \\
\text{Amplitude} &= \frac{E_0}{\sqrt{L^2\Omega^2 + R^2}}
\end{aligned}
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